

# Operator identities corresponding to inverse problems

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## Abstract

The structured operators and corresponding operator identities, which appear in inverse problems for the self-adjoint and skew-self-adjoint Dirac systems with rectangular potentials, are studied in detail. In particular, it is shown that operators with the close to displacement kernels are included in this class. A special case of positive and factorizable operators is dealt with separately.

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## 1 Introduction

An operator  $S$  with a difference kernel was used by M.G. Krein to solve the inverse spectral problem for the self-adjoint Dirac-type system in his classical work [20], see also [4, 6, 18, 24, 34] and references therein. Following papers [28–30] on the method of operator identities and its applications to inverse spectral problems for canonical systems, various other systems were treated in the same way using other operators satisfying somewhat different operator identities (see, e.g., [9–11, 23, 25, 26, 32, 33]).

In particular, in this paper we study operators, which are necessary to recover the self-adjoint Dirac system

$$\frac{d}{dx}y(x, z) = i(zj + jV(x))y(x, z), \quad x \geq 0; \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad (1.1)$$

where  $I_{m_i}$  is the  $m_i \times m_i$  identity matrix,  $V = \{V_{i,j}\}_{i,j=1}^2$ ,  $V_{11} = 0$ ,  $V_{22} = 0$ ,  $V_{12} = V_{21}^* = v$ , and the  $m_1 \times m_2$  block  $v(x)$  of  $V(x)$  is called the potential. The skew-self-adjoint analog of system (1.1) has the form

$$\frac{d}{dx}y(x, z) = (izj + jV(x))y(x, z), \quad x \geq 0. \quad (1.2)$$

Systems (1.1) and (1.2) are auxiliary linear systems for various important integrable coupled, multicomponent, and matrix wave equations (see, e.g., [1–3, 37] and references therein).

The direct problem for system (1.1) was treated in [8], and the existence of the  $m_2 \times m_1$  non-expansive Weyl function was proved. To solve the inverse problem and recover system (1.1) from its Weyl function, the study of operators  $S$ , which satisfy operator identities of the form

$$AS - SA^* = i\Pi j\Pi^*, \quad A, S \in B(L_{m_2}^2(0, l)), \quad A = -i \int_0^x \cdot dt; \quad (1.3)$$

$$\Pi := [\Phi_1 \quad \Phi_2] \in B(\mathbb{C}^m, L_{m_2}^2(0, l)), \quad m := m_1 + m_2, \quad (1.4)$$

$$\Phi_i \in B(\mathbb{C}^{m_i}, L_{m_2}^2(0, l)), \quad (\Phi_1 f)(x) = \Phi_1(x)f, \quad \Phi_2 f = I_{m_2} f \equiv f, \quad (1.5)$$

is required. Here  $\mathbb{C}$  stands for the complex plain,  $B(\mathbf{H}_1, \mathbf{H}_2)$  denotes the class of bounded linear operators acting from the space  $\mathbf{H}_1$  into the space  $\mathbf{H}_2$ ,  $B(\mathbf{H})$  is the class of bounded linear operators, which map  $\mathbf{H}$  into itself, and  $\Phi_1(x)$  is an  $m_2 \times m_1$  matrix function. The notation  $I$  will be used for the identity operator.

The related operator identities, which appear in the case of skew-self-adjoint system (1.2), have the form (see [10, 23] for the case that  $m_1 = m_2$ )

$$i(AS - SA^*) = \Pi\Pi^*, \quad A, S \in B(L_{m_2}^2(0, l)), \quad A = -i \int_0^x \cdot dt, \quad (1.6)$$

where  $\Pi$  is given by formulas (1.4) and (1.5).

What are often referred to as structured operators (that satisfy operator identities) are also of independent interest. For applications of structured operators to probability theory and other domains see [7, 14, 16, 31, 35, 36] and various references therein. In particular, we show that, for the case that  $\Phi_1(x)$  is continuously differentiable, the operators  $S$  satisfying (1.3) have

*close to displacement kernels.* Operators with close to displacement kernels were considered in [17] (see also [13, Section 2.4] and references therein) in connection with slightly non-homogenous processes and an algebra generated by Toeplitz operators. We also derive explicit inversion formulas for our operators. Explicit inversion formulas for convolution integral operators on a finite interval are presented in [12].

## 2 Operator identity: the case of self-adjoint Dirac system

We fix some  $0 < l < \infty$  and consider an operator  $S \in B(L_{m_2}^2(0, l))$ :

$$(Sf)(x) = (I_{m_2} - \Phi_1(0)\Phi_1(0)^*)f(x) - \int_0^l s(x, t)f(t)dt, \quad (2.1)$$

$$s(x, t) := \int_0^{\min(x, t)} \Phi_1'(x - \zeta)\Phi_1'(t - \zeta)^*d\zeta + \begin{cases} \Phi_1'(x - t)\Phi_1(0)^*, & x > t; \\ \Phi_1(0)\Phi_1'(t - x)^*, & t > x. \end{cases} \quad (2.2)$$

As mentioned in the Introduction, for the case that  $\Phi_1(x)$  is continuously differentiable, the kernel  $s$  of the form (2.2) is called a "close to displacement" kernel [13, 17].

**Proposition 2.1** *Let  $\Phi_1(x)$  be an  $m_2 \times m_1$  matrix function, which is boundedly differentiable on the interval  $[0, l]$ . Then the operator  $S$ , which is given by (2.1) and (2.2), satisfies the operator identity (1.3), where  $\Pi := [\Phi_1 \quad \Phi_2]$  is expressed via formulas (1.4) and (1.5).*

To proceed with the proof we need Proposition 3.2 from [10], the formulation and proof of which are valid also for rectangular matrix functions  $k$  and  $\tilde{k}$  (though it is not stated in [10] directly). We rewrite Proposition 3.2:

**Lemma 2.2** [10] *Let  $\Phi(x)$  and  $\hat{\Phi}(x)$  be, respectively,  $m_2 \times m_1$  and  $m_1 \times m_2$  matrix functions, which are boundedly differentiable on the interval  $[0, l]$  and satisfy equalities  $\Phi(0) = 0$ ,  $\hat{\Phi}(0) = 0$ . Then the operator  $S$ , which is given by*

$$Sf = -\frac{1}{2} \int_0^l \int_{|x-t|}^{x+t} \Phi'\left(\frac{\xi + x - t}{2}\right) \hat{\Phi}'\left(\frac{\xi + t - x}{2}\right) d\xi f(t) dt, \quad (2.3)$$

satisfies the operator identity

$$AS - SA^* = i\Phi(x) \int_0^l \widehat{\Phi}(t) \cdot dt. \quad (2.4)$$

The scalar subcase  $m_2 = 1$  of Lemma 2.2 was earlier dealt with in [19].

*Proof* of Proposition 2.1. Rewrite (2.1) as

$$\begin{aligned} S &= \sum_{i=1}^4 S_i, \quad (S_1 f)(x) = (I_{m_2} - \Phi_1(0)\Phi_1(0)^*)f(x), \\ S_2 &= - \int_0^x \Phi_1'(x-t)\Phi_1(0)^* \cdot dt, \quad S_3 = - \int_x^l \Phi_1(0)\Phi_1'(t-x)^* \cdot dt, \\ S_4 &= - \int_0^l \int_0^{\min(x,t)} \Phi_1'(x-\zeta)\Phi_1'(t-\zeta)^* d\zeta \cdot dt. \end{aligned} \quad (2.5)$$

It is immediately clear that

$$AS_1 - S_1A^* = i(\Phi_1(0)\Phi_1(0)^* - I_{m_2}) \int_0^l \cdot dt. \quad (2.6)$$

By changing of order of integration and integrating by parts we easily get

$$AS_2 - S_2A^* = i(\Phi_1(x) - \Phi_1(0))\Phi_1(0)^* \int_0^l \cdot dt, \quad (2.7)$$

$$AS_3 - S_3A^* = i\Phi_1(0) \int_0^l (\Phi_1(t) - \Phi_1(0))^* \cdot dt. \quad (2.8)$$

Because of (2.5)-(2.7), it remains to show that

$$AS_4 - S_4A^* = i(\Phi_1(x) - \Phi_1(0)) \int_0^l (\Phi_1(t) - \Phi_1(0))^* \cdot dt \quad (2.9)$$

to prove (1.3). Finally, after substitution

$$\xi = x + t - 2\zeta, \quad \Phi(x) = \Phi_1(x) - \Phi_1(0), \quad \widehat{\Phi}(t) = (\Phi_1(t) - \Phi_1(0))^*,$$

it follows that operator  $S$  in Lemma 2.2 equals  $S_4$ , and formula (2.4) yields (2.9). Thus, (1.3) is proved.  $\square$

The useful proposition below is a special case of Theorem 3.1 in [27] (and a simple generalization of a subcase of scalar Theorem 1.3 [31, p. 11]).

**Proposition 2.3** *Suppose an operator  $T \in B(L_{m_2}^2(0, l))$  satisfies the operator identity*

$$TA - A^*T = i \int_0^l Q(x, t) \cdot dt, \quad Q(x, t) = Q_1(x)Q_2(t), \quad (2.10)$$

where  $Q$ ,  $Q_1$ , and  $Q_2$  are  $m_2 \times m_2$ ,  $m_2 \times p$ , and  $p \times m_2$  ( $p > 0$ ) matrix-functions, respectively. Then  $T$  has the form

$$Tf = \frac{d}{dx} \int_0^l \frac{\partial}{\partial t} \Upsilon(x, t) f(t) dt, \quad (2.11)$$

where  $\Upsilon$  is absolutely continuous in  $t$  and

$$\Upsilon(x, t) := -\frac{1}{2} \int_{x+t}^{2l-|x-t|} Q_1\left(\frac{\xi+x-t}{2}\right) Q_2\left(\frac{\xi-x+t}{2}\right) d\xi. \quad (2.12)$$

In fact, even the scalar version of Proposition 2.3 could be used to show the uniqueness of the solution  $S$  of (1.3).

**Corollary 2.4** *The operator  $S = 0$  is the unique operator  $S \in B(L_{m_2}^2(0, l))$ , which satisfies the operator identity  $AS - SA^* = 0$ .*

*Proof.* We prove by contradiction. Let  $S_0 \neq 0$  ( $S_0 \in B(L_{m_2}^2(0, l))$ ) satisfy the identity  $AS_0 - S_0A^* = 0$ . From definition of  $A$  in (1.3), we have

$$UAU = A^*, \quad UA^*U = A \quad \text{for} \quad (Uf)(x) := \overline{f(l-x)}. \quad (2.13)$$

It follows directly from the identity  $AS_0 - S_0A^* = 0$  and equality (2.13) that

$$T_0A - A^*T_0 = 0 \quad \text{for} \quad T_0 := US_0U \neq 0, \quad (2.14)$$

where  $T_0 \in B(L_{m_2}^2(0, l))$ . Then, Proposition 2.3 and formula (2.14) imply  $T_0 = 0$  and we arrive at a contradiction.  $\square$

Proposition 2.1 and Corollary 2.4 yield the following result.

**Theorem 2.5** *Let  $\Phi_1(x)$  be an  $m_2 \times m_1$  matrix function, which is boundedly differentiable on the interval  $[0, l]$ . Then the operator  $S$ , which is given by (2.1) and (2.2), is the unique solution of the operator identity (1.3), where  $\Pi = [\Phi_1 \quad \Phi_2]$  is expressed via formulas (1.4) and (1.5).*

Note that the operator  $S^*$  satisfies (1.3) (or (1.6)) together with  $S$ , and so  $S = S^*$  is immediate from the uniqueness of the solution of the corresponding operator identity.

### 3 Operator identity: the case of skew-self-adjoint Dirac system

Let

$$S = 2I - \check{S}, \quad A\check{S} - \check{S}A^* = i\Pi_j\Pi^*. \quad (3.1)$$

In view of (1.5) and (1.6) we have

$$i(A - A^*) = \Phi_2\Phi_2^*. \quad (3.2)$$

Therefore, the first equality in (3.1) yields equivalence between the second equality in (3.1) and identity (1.6). In other words, we can rewrite Theorem 2.5 in the following way.

**Theorem 3.1** *Let  $\Phi_1(x)$  be an  $m_2 \times m_1$  matrix function, which is boundedly differentiable on the interval  $[0, l]$ . Then the operator  $S$ , which is given by*

$$(Sf)(x) = (I_{m_2} + \Phi_1(0)\Phi_1(0)^*)f(x) + \int_0^l s(x, t)f(t)dt \quad (3.3)$$

*and (2.2), is the unique solution of the operator identity (1.6), where  $\Pi$  is expressed via formulas (1.4) and (1.5).*

The case of positive operators  $S$  is of interest, as these are operators that appear in inverse (and many other) problems.

**Proposition 3.2** *The operators  $S$  considered in Theorem 3.1 are always strictly positive. Furthermore, the inequality  $S \geq I$  holds.*

**Proof.** It suffices to show that the inequalities  $\mathcal{S}_\varepsilon \geq 0$ , where  $\mathcal{S}_\varepsilon$  is given by

$$(\mathcal{S}_\varepsilon f)(x) = (\varepsilon I_{m_2} + \Phi_1(0)\Phi_1(0)^*)f(x) + \int_0^l s(x, t)f(t)dt, \quad (3.4)$$

hold for all  $0 < \varepsilon < 1$ . For that purpose we note that  $\mathcal{S}_\varepsilon = S - (1 - \varepsilon)I$ . Therefore identities (1.6) and (3.2) lead us to the formula

$$i(A\mathcal{S}_\varepsilon - \mathcal{S}_\varepsilon A^*) = \Phi_1\Phi_1^* + \varepsilon\Phi_2\Phi_2^* \geq 0, \quad (3.5)$$

that is, the operator  $A^*$  is  $\mathcal{S}_\varepsilon$ -dissipative.

Next, we will use several statements from [5], where earlier results (results on operators in the space  $\Pi_\varkappa$  from [21,22]) are developed for the case that we are interested in. Because of [5, statement 9°] we see that  $A^* \ker \mathcal{S}_\varepsilon \subseteq \ker \mathcal{S}_\varepsilon$ . Since the integral part of  $\mathcal{S}_\varepsilon$  is a compact operator, we derive that  $\ker \mathcal{S}_\varepsilon$  is finite-dimensional. However,  $A^*$  does not have eigenvectors and finite-dimensional invariant subspaces. Therefore, we get  $\ker \mathcal{S}_\varepsilon = 0$ , and so  $\mathcal{S}_\varepsilon$  admits the representation

$$\mathcal{S}_\varepsilon = -KJK, \quad K > 0, \quad J = P_1 - P_2 \quad (P_i, K, K^{-1} \in B(L_{m_2}^2(0, l))), \quad (3.6)$$

where  $P_1$  and  $P_2$  are orthoprojectors,  $P_1 + P_2 = I$ . Furthermore, since  $\varepsilon > 0$  and the integral part of  $\mathcal{S}_\varepsilon$  is a compact operator, we see that  $P_1$  is a finite-dimensional orthoprojector. In other words,  $J$  determines some space  $\Pi_\varkappa$ , where  $\varkappa < \infty$  is the dimension of  $\text{Im } P_1$ . According to (3.5) and (3.6) the operator  $-KA^*K^{-1}$  is  $J$ -dissipative. From [5, Theorem 1] we see that there is a  $\varkappa$ -dimensional invariant subspace of  $-KA^*K^{-1}$  (i.e., there is a  $\varkappa$ -dimensional invariant subspace of  $A^*$ ), which leads us to  $\varkappa = 0$  and  $J = -I$ . Now, the inequality  $\mathcal{S}_\varepsilon \geq 0$  follows directly from the first relation in (3.6).  $\square$

## 4 Families of positive operators

In this section we consider different values of  $l$  simultaneously, and so the operator  $S \in B(L_{m_2}^2(0, l))$ , which is given by (2.1), will be denoted by  $S_l$  with index "l" below (correspondingly,  $A$  will be denoted by  $A_l$ , and  $\Pi$  by  $\Pi_l$ ). Next, introduce an orthoprojector  $P_r$  ( $r \leq l$ ) from  $L_{m_2}^2(0, l)$  on  $L_{m_2}^2(0, r)$  such that

$$(P_r f)(x) = f(x) \quad (0 < x < r), \quad f \in L_{m_2}^2(0, l). \quad (4.1)$$

Clearly, for  $\hat{l} < l$  we have

$$A_{\hat{l}} = P_{\hat{l}} A P_{\hat{l}}^*, \quad S_{\hat{l}} = P_{\hat{l}} S_l P_{\hat{l}}^*. \quad (4.2)$$

The case of positive operators  $S_l$ , which satisfy (1.3) (as well as positive operators  $S$ , which satisfy (1.6) and were dealt with in Section 3), is of

special interest. Such operators are invertible and admit the factorization

$$S_l^{-1} = E_{\Phi,l}^* E_{\Phi,l}, \quad E_{\Phi,l} = I + \int_0^x E_{\Phi}(x, t) \cdot dt \in B(L_2^{m_2}(0, l)). \quad (4.3)$$

More precisely, the following statements hold.

**Proposition 4.1** *Let  $\Phi_1(x)$  be an  $m_2 \times m_1$  matrix function, which is boundedly differentiable on the interval  $[0, l]$  and satisfies the inequality*

$$(I_{m_2} - \Phi_1(0)\Phi_1(0)^*) > 0. \quad (4.4)$$

*Furthermore, let operators  $S_r$  of the form (2.1), where  $s$  is expressed via (2.2), be boundedly invertible for all  $0 < r \leq l$ . Then the operators  $S_r$  are strictly positive (i.e.,  $S_r > 0$ ).*

**Proof.** Since (4.4) holds and operators  $S_r$  are given by (2.1), where  $l$  is substituted by  $r$ , we have  $S_r > 0$  for small values of  $r$ . We proceed by negation and suppose that some operators  $S_r$  are not strictly positive. Then there is a value  $0 < r_0 < l$  such that  $S_{r_0} > 0$  and the inequality does not hold for all  $r > r_0$ . This is impossible, since the inequality  $S_{r_0} > 0$  and formula (4.4) imply  $S_{r_0+\varepsilon} > 0$  for small values of  $\varepsilon$ .  $\square$

**Theorem 4.2** *Let the matrix function  $\Phi_1(x)$  and operators  $S_l$ , which are expressed via  $\Phi_1$  in (2.1), be such that  $\Phi_1$  is boundedly differentiable on each finite interval  $[0, l]$  and satisfies equality  $\Phi_1(0) = 0$ , while the operators  $S_l$  are boundedly invertible for all  $0 < l < \infty$ . Then the operators  $S_l^{-1}$  admit factorizations (4.3), where  $E_{\Phi}(x, t)$  is continuous with respect to  $x, t$  and does not depend on  $l$ . Furthermore, all the factorizations (4.3) with continuous  $E_{\Phi}(x, t)$  are unique.*

**Proof.** Since  $\Phi_1(0) = 0$ , formula (2.1) takes the form

$$S_l = I - \int_0^l s(x, t) \cdot dt, \quad s(x, t) = \int_0^{\min(x, t)} \Phi_1'(x - \zeta) \Phi_1'(t - \zeta)^* d\zeta. \quad (4.5)$$

Because of (4.5) we see that the kernel  $s(x, t)$  of  $S_l$  is continuous. Hence, we can apply the factorization "result 2" from [15, pp. 185-186]. It follows



that operators  $S_l^{-1}$  admit upper-lower triangular factorizations, where the kernels of the corresponding triangular operators are continuous. Taking into account the equality  $S_l = S_l^*$  (i.e.,  $S_l^{-1} = (S_l^{-1})^*$ ), we use formulas (7.8) and (7.9) from [15, p. 186] to show that the upper triangular factor of  $S_l^{-1}$  is adjoint to the lower triangular factor, that is, formula (4.3) holds. Moreover, formulas (7.8) and (7.9) from [15, p. 186] imply that  $E_\Phi(x, t)$  in (4.3) does not depend on  $l$ . The uniqueness of the factorization (4.3) is immediate from the uniqueness of the upper-lower triangular factorization of  $I$  which, in turn, easily follows from the relations for kernels of the factors yielded by the factorization formula for  $I$ .  $\square$

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